ANISOTROPIC MODEL OF AN EMITTING ELECTRIC ARC

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An analytical procedure is proposed for obtaining exponential dimensionless equations to calculate the characteristics of an electric arc in a plasmatron channel that is based on using an exponential approximation of the temperature dependence of electric conductivity with different exponents for longitudinal and transverse coordinates.

Solving rather complex nonlinear problems of calculating the characteristics of electric arcs in plasmatron channels by numerical methods generally does not cause any fundamental difficulties. To bring the problem closer to real processes, one is led to employ numerous variable parameters. However, it is rather difficult to arrange calculation results into a certain system and to find hidden relationships between these variables. The methods of the approximate similarity theory permit the employment of a much smaller number of generalized variables, but obtaining dimensionless dependences involves expensive experiments. Furthermore, the semiempirical character of generalized formulas restricts the possibilities for their subsequent analysis.

Use of analytical methods that permit us, to a great extent, to circumvent the indicated difficulties is strongly restricted by the requirement on nonlinearity of the equations. A more efficient use of the analytical methods involves difficulties in solving nonlinear problems. One way to surmount them that leads to generalized exponential formulas is discussed below.

We write the system of equations for an electric arc in a cylindrical channel with longitudinal gas flow

$$\rho C_p V_z \frac{\partial T}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left(r\lambda \frac{\partial T}{\partial r} \right) + \sigma E_z^2 - Q \,, \tag{1}$$

$$G = 2\pi \int\limits_0^R \rho v_z r dr \,, \tag{2}$$

$$I = 2\pi \int\limits_0^R \sigma E_z r dr \,. \tag{3}$$

In truncated energy Eq. (1), we ignore heat conduction in the longitudinal direction, turbulent transfer, convection in the radial direction, the kinetic energy of ordered motion, viscous dissipation, and the radial component of Joule dissipation. The radiation is taken account of in a volume degree of approximation. In addition, below we assume C_p/λ = const and ρV_z = const and divide the space in the radial direction into conducting ($\sigma \neq 0$) and nonconducting ($\sigma = 0$) bands. By introducing the function of thermal conductivity $S = \int\limits_0^T \lambda dT$ as well as $\bar{r} = r/R$, $\bar{z} = z/R$, $\Delta S = S - S_*$, $a = GC_p/\pi R\lambda$, $\sigma = \sigma_0(\Delta S/\Delta S_0)^{n_\sigma}$, and $Q = Q_0(\Delta S/\Delta S_0)$ we transform Eq. (1) for the conducting band using (2) and (3) to the form

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$$a\frac{\partial \Delta S_{\mathrm{I}}}{\partial \bar{z}} = \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial \Delta S_{\mathrm{I}}}{\partial \bar{r}}\right) + \frac{G_0\,(\Delta S_{\mathrm{I}}/\Delta S_0)^{n_\sigma} I^2}{(2\pi R)^2\left[\int\limits_0^{\bar{r}_*} \sigma_0\,(\Delta S_{\mathrm{I}}/\Delta S_0)^{n_\sigma}\bar{r}d\bar{r}\right]^2} - Q_0\,(\Delta S_{\mathrm{I}}/\Delta S_0)\,R^2. \tag{4}$$

We solve Eq. (4) for the boundary conditions

$$\Delta S_{\mathrm{I}}\,(\bar{r},\,0) = 0;\quad \Delta S_{\mathrm{I}}\,(\bar{r}_*,\,\bar{z});\quad \Delta S_{\mathrm{I}}\,(0,\,\bar{z}) \neq \infty. \tag{5}$$

The similar equation and the boundary conditions for the nonconducting band are written as follows:

$$a\frac{\partial \Delta S_{\mathrm{II}}}{\partial \bar{z}} = \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial \Delta S_{\mathrm{II}}}{\partial \bar{r}}\right); \tag{6}$$

$$\Delta S_{\mathrm{II}}\,(\bar{r}_*,\,\bar{z}) = 0;\quad \Delta S_{\mathrm{II}}\,(1,\,\bar{z}) = \Delta S_{\mathrm{I}};\quad \Delta S_{\mathrm{II}}\,(\bar{r},\,0) = \Delta S_{\mathrm{I}}. \tag{7}$$

We solve Eqs. (4) and (6) by separating the variables. We represent the solution of (4) as the product of two functions $\Delta S_{\mathrm{I}} = \theta(\bar{r})\vartheta(\bar{z})$. By omitting now the bar over $\bar{p}$ and $\bar{z}$ we have

$$a\theta\frac{\partial\vartheta}{\partial z} = \vartheta\left(\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r}\right) + + \frac{\theta^{n_\sigma} I^2}{(2\pi R)^2\,\sigma_0\,(\vartheta/\Delta S_0)^{n_\sigma}\left(\int\limits_0^{n_*}\theta^{n_\sigma}rdr\right)^2} - b_Q R^2\theta\vartheta. \tag{8}$$

To be able to separate the variables, we substitute the linear function $\kappa\theta$ for θ^{n_σ} in the numerator of the Joule component of Eq. (8). With this procedure we perform a separate approximation by different coordinates of the dependence $\sigma = f(\Delta S)$, introducing an artificial (approximation) anisotropy for the electrical conductivity of the plasma. Along the longitudinal coordinate the electrical conductivity changes with S according to a power law and along the radius to a linear law. A partial solution of (8) with condition (5) and in view of $\theta^{n_\sigma} = \kappa\theta$ can be obtained as

$$\Delta S_{\mathrm{I}}\,(r,\,z) = \Delta S_{00}\,[1 - \exp(-Bz)]^{\frac{1}{n_\sigma+1}}\,J_0\,(\mu_1 r/r_*), \tag{9}$$

where μ_1 is the first root of the characteristic equation $J_0(kr_*) = 0$,

$$\Delta S_{00} = A^{\frac{1}{n_\sigma+1}}\,C^{\frac{1-n_\sigma}{1+n_\sigma}} \tag{10}$$

is the value of S on the axis as $Bz \to \infty$,

$$A = \frac{I^2\kappa\Delta S_0^{n_\sigma}}{k_\sigma^2\,(2\pi R)^2\,\sigma_0 r_*^2 J_0^2\,(\mu_1)\,(1 + b_Q R^2 r_*^2/\mu_1^2)}, \tag{11}$$

$$B = \frac{(n_\sigma + 1)\,\mu_1^2\pi R\lambda\,(1 + b_Q R^2 r_*^2/\mu_1^2)}{r_*^2 G C_p}. \tag{12}$$

The constant C_{00} in (10) in the interrelationship with κ in A will be determined below.

The coefficient k_σ is obtained in calculating the integral $\int\limits_0^{r_*} \theta^{n_\sigma} r dr$ in (8), where $\theta = C_{00} J_0(\mu_1 r / r_*)$ is the solution of the equation for $\theta(r)$ upon separation of the variables. By resorting to the expansion of θ^{n_σ} into a series at the point $\theta = 1$ and confining ourselves to three terms of the expansion, we obtain an expression for determining k_σ:

$$k_\sigma = \frac{(1 - n_\sigma)\mu_1}{2J_1(\mu_1)} + n_\sigma + \frac{n_\sigma(n_\sigma - 1)\mu_1 J_1(\mu_1)}{4} -$$

$$- n_\sigma(n_\sigma - 1) + \frac{n_\sigma(n_\sigma - 1)\mu_1}{4J_1(\mu_1)}. \tag{13}$$

Expression (9), when $n_\sigma = 1$ and $b_Q = 0$, coincides with G. Stain's solution [1] and, when $n_\sigma = 1$ and $Bz \to \infty$, $b_Q = 0$, with G. Mekker's solution [2].

The solution of (6) with conditions (7) can be represented as a series [3]:

$$\Delta S_{\text{II}}(r, z) = -\Delta S_1 \left[\frac{\ln \frac{r}{r_*}}{\ln r_*} - \pi \sum_{n=1}^{\infty} \left(1 - \frac{J_0(\alpha_n)}{J_0(\alpha_n) - J_0(\alpha_n m)} \right) \times \frac{J_0(\alpha_n) U_0(\alpha_n m r)}{J_0(\alpha_n) + J_0(\alpha_n m)} \exp\left(-\frac{\alpha_n^2}{d} z \right) \right], \tag{14}$$

where

$$U_0(\alpha_n m r) = J_0(\alpha_n m r) Y_0(\alpha_n m) - J_0(\alpha_n m) Y_0(\alpha_n m r); \tag{15}$$

α_n are roots of the characteristic equation

$$J_0(\alpha_n) Y_0(\alpha_n m) - J_0(\alpha_n m) Y_0(\alpha_n) = 0. \tag{16}$$

We determine the boundary of the conducting band from the condition

$$\left. \frac{\partial \Delta S_{\text{I}}}{\partial r} \right|_{r=r_*} = \left. \frac{\partial \Delta S_{\text{II}}}{\partial r} \right|_{r=r_*}. \tag{17}$$

By differentiating (9) and (14) and by substituting into (17) we obtain an expression for determining r_*

$$\left(\frac{C_{00}^{\frac{1-n_\sigma}{1+n_\sigma}} \mu_1 J_1(\mu_1) \Delta S_0}{\Delta S_1} \right) \times \left\{ \frac{\kappa I^2}{k_\sigma^2 (2\pi R)^2 \sigma_0 \Delta S_0 J_1(\mu_1) r_*^2 (1 + b_Q R^2 r_*^2 / \mu_1^2)} [1 - \exp(-Bz)] \right\}^{\frac{1}{n_\sigma + 1}} =$$

$$= \frac{1}{\ln r_*} - 2 \sum_{n=1}^{\infty} \frac{1}{k_n^2 - 1} \exp\left(-\frac{\alpha_n^2}{d} z \right), \tag{18}$$

where

$$k_n = \frac{J_0(\alpha_n)}{J_0(\alpha_n m)} = \frac{Y_0(\alpha_n)}{Y_0(\alpha_n m)}. \tag{19}$$

To determine the constant C_{00}, we need to consider the case of $Bz \to \infty$ and $\alpha_n^2 z / a \to \infty$, i.e., the characteristics of a nonconsumable arc or an arc on the portion of developed heat transfer (the asymptotic portion), Eq. (1) for which at $b_Q = 0$ appears as

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\Delta S}{dr}\right)+J_zE_zR^2=0\,. \tag{20}$$

We multiply (20) by $2\pi Rdr$ and integrate once between 0 and 1, after which we obtain

$$E_zI=2\pi Rq_1\,, \tag{21}$$

where

$$q_1=-\frac{1}{R}\frac{d\Delta S}{dr}\bigg|_{r=1}\,. \tag{22}$$

Expression (21) holds for any dependence $\sigma(S)$. To determine C_{00}, we resort to (14), (18), (21), (22), and the expression for E_z that can be obtained from (3) in view of (9), with the conditions $Bz\to\infty$ and $\alpha_n^2 z/a\to\infty$:

$$E_z=\frac{\mu_1 I}{C_{00}^{\frac{n_\sigma(1-n_\sigma)}{1+n_\sigma}}\,2\pi R^2r_*^2J_1(\mu_1)\,A^{-\frac{n_\sigma}{n_\sigma+1}}}\,. \tag{23}$$

We obtain

$$C_{00}=\left(\frac{k_\sigma}{\kappa}\right)^{\frac{1}{1-n_\sigma}}\,. \tag{24}$$

By employing (19) and (24) from (3), (9), (14), (18), and (22) we obtain the final dimensionless expressions for determining the radius of the conducting band, the distribution of the thermal conductivity function, the electric field strength, and the heat flux on the wall. We write them respectively in expanded form:

$$\left\{\frac{1}{4\pi^2k_\sigma J_1^2(\mu_1)\,r_*^2\,(1+Q_0R^2r_*^2/\Delta S_0\,\mu_1^2)}\,\frac{I^2}{R^2\sigma_0\Delta S_0}\times\right.$$

$$\left.\times\left[1-\exp\left(-\frac{(n_\sigma+1)\,\mu_1^2\,(1+Q_0R^2r_*^2/\Delta S_0\,\mu_1^2)}{r_*^2}\,\frac{\pi R\lambda}{GC_p}\,z\right)\right]\right\}^{\frac{1}{n_\sigma+1}}=$$

$$=\frac{1}{J_1(\mu_1)\,\mu_1}\left(-\frac{\Delta S_1}{\Delta S_0}\right)\left[\frac{1}{\ln(1/r_*^2)}+2\sum_{n=1}^{\infty}\frac{1}{k_n^2-1}\exp\left(-\alpha_n^2\,\frac{\pi R\lambda}{GC_p}\,z\right)\right]; \tag{25}$$

$$\frac{\Delta S_{\mathrm{I}}(r,\,z)}{\Delta S_0}=\left\{\frac{1}{4\pi^2k_\sigma J_1^2(\mu_1)\,r_*^2\,(1+Q_0R^2r_*^2/\Delta S_0\,\mu_1^2)}\times\right.$$

$$\left.\times\frac{I^2}{R^2\sigma_0\Delta S_0}\left[1-\exp\left(-\frac{(n_\sigma+1)\,\mu_1^2\,(1+Q_0R^2r_*^2/\Delta S_0\,\mu_1^2)}{r_*^2}\,\frac{\pi R\lambda}{GC_p}\,z\right)\right]\right\}^{\frac{1}{n_\sigma+1}}J_0\left(\mu_1\frac{r}{r_*}\right); \tag{26}$$

$$\frac{\Delta S_{II}(r, z)}{\Delta S_0} = -\frac{\Delta S_1}{\Delta S_0}\left[\frac{\ln(r/r_*)}{\ln r_*} + \pi\sum_{n=1}^{\infty}\frac{k_n}{k_n^2 - 1}U_0(\alpha_n r/t_*)\exp\left(-\alpha_n^2\frac{\pi R\lambda}{GC_p}z\right)\right]; \tag{27}$$

$$\frac{E_z R^2\sigma_0}{I} = \frac{\mu_1}{2\pi k_\sigma J_1(\mu_1) r_*^2}\times\left\{\frac{1}{4\pi^2 k_\sigma J_1^2(\mu_1) r_*^2(1 + Q_0R^2r_*^2/\Delta S_0\mu_1^2)}\frac{I^2}{R^2\sigma_0\Delta S_0}\times\right.$$

$$\left.\times\left[1 - \exp\left(-\frac{(n_\sigma + 1)\mu_1(1 + Q_0R^2r_*^2/\Delta S_0\mu_1^2)}{r_*^2}\frac{\pi R\lambda}{GC_p}z\right)\right]\right\}^{-\frac{n_\sigma}{n_\sigma+1}}; \tag{28}$$

$$\frac{q_1 R}{\Delta S_0} = -\frac{\Delta S_1}{\Delta S_0}\left[\frac{1}{\ln(1/r_*)} + 2\sum_{n=1}^{\infty}\frac{k_n}{k_n^2 - 1}\exp\left(-\alpha_n^2\frac{\pi R\lambda}{GC_p}z\right)\right]. \tag{29}$$

The complex $P_0 = I^2/R^2\sigma_0\Delta S_0$ in (25)-(29) is a modified, as applied to the arc, Pomerantsev number [3]. The generalized function $\Pi_E = E_zR^2\sigma_0/I$ is a modification of the known similarity number $\Pi_U = UD\sigma_0/I$ [4] when the electric field strength is determined rather than the potential difference on the arc. The complex variable $\Pi_q = q_1R/\Delta S_0$ at $\lambda = \lambda_0 = \text{const}$ with $q_1 = \alpha\Delta T_0$ substituted transforms to the known Nusselt number $\mathrm{Nu} = \alpha R/\lambda_0$. Furthermore, the number $K_Q = Q_0R^2/\Delta S_0$, reflecting the radiant-to-convective transfer relation, and the parametric number $K_S = -\Delta S_1/\Delta S_0$ enter into the above expressions. In (25)-(29), there is also the known Pecklet number: $\mathrm{Pe} = GC_{p0}/\pi R\lambda_0$.

By employing the symbols Po, Pe, K_Q, K_S, Π_E, and Nu we can represent expressions (25)-(29) as dimensionless equations with numerical coefficients:

$$\left\{\frac{\mathrm{Po}}{10.6k_\sigma r_*^2(1 + 0.17K_Qr_*^2)}\times\left[1 - \exp\left(-\frac{5.78(1 + 0.17K_Qr_*^2)}{\mathrm{Pe}\, r_*^2}z\right)\right]\right\}^{\frac{1}{n_\sigma+1}} =$$

$$= 0.8K_s\left[\frac{1}{\ln(1/r_*)} + 2\sum_{n=1}^{\infty}\frac{k_1}{k_n^2 - 1}\exp\left(-\frac{\alpha_n^2}{\mathrm{Pe}}z\right)\right]; \tag{30}$$

$$\frac{\Delta S_I(r, z)}{\Delta S_0} = \left\{\frac{\mathrm{Po}}{10.6k_\sigma r_*^2(1 + 0.17K_Qr_*^2)}\times\right.$$

$$\left.\times\left[1 - \exp\left(-\frac{5.78(n_\sigma + 1)(1 + 0.17K_Qr_*^2)}{\mathrm{Pe}\, r_*^2}z\right)\right]\right\}^{\frac{1}{n_\sigma+1}} J_0\left(\mu_1\frac{r}{r_*}\right); \tag{31}$$

$$\frac{\Delta S_{II}(r, z)}{\Delta S_0} = K_S\left[\frac{\ln(r/r_*)}{\ln r_*} + 3.14\sum_{n=1}^{\infty}\frac{k_n}{k_n^2 - 1}U_0(\alpha_n r/r_*)\exp\left(-\frac{\alpha_n^2}{\mathrm{Pe}}z\right)\right]; \tag{32}$$

$$\Pi_E = \frac{0.74}{k_\sigma r_*^2} \left\{ \frac{\text{Po}}{10.6 k_\sigma r_*^2 (1 + 0.17 K_Q r_*^2)} \times \left[1 - \exp\left(- \frac{5.78 (n_\sigma + 1)(1 + 0.17 K_Q r_*^2)}{\text{Pe}\, r_*^2} z \right) \right] \right\}^{-\frac{n_\sigma}{n_\sigma + 1}} ; \quad (32)$$

$$\text{Nu} = K_S \left[\frac{1}{\ln (1/r_*)} + 2 \sum_{n=1}^{\infty} \frac{k_n}{k_n^2 - 1} \exp\left(- \frac{\alpha_n^2}{\text{Pe}} z \right) \right] . \quad (34)$$

It is evident that the dimensionless radius of the conducting zone, thermal conductivity function, electric field strength, and heat flux into the wall are the functions of the numbers Po, Pe, K_Q, and K_S. Calculations using the dependences obtained are complicated in the general case by the transcendence of Eq. (30) for determining the conducting band radius. However, for some particular cases the expressions are substantially simplified. In [5], dimensionless equations are given for the asymptotic portion of an arc in a channel (a nonconsumable or weakly ventilated arc). Comparison of the calculation with experiment made in this work for two gases (for argon and air) in physical and generalized coordinates showed their satisfactory (±26 %) agreement.

The proposed method enables us to obtain dimensionless dependences for calculating the characteristics of electric arcs without using an experiment, which up to now has been done only by generalizing a great body experimental data.

NOTATION

ρ, density; C_p, specific heat at constant pressure; σ, electrical conductivity; λ, thermal conductivity; V, velocity; T, temperature; E, electric-field strength; r, z, radial and longitudinal coordinates; $S = \int_0^T \lambda dT$, thermal conductivity function; G, gas flow rate; Q, bulk radiation density; R, D, radius and diameter of channel; I, strength of current; $\bar{r} = r/R$; $\bar{z} = z/R$; $m = R/r_* = 1/r_*$; $\Delta S = S - S_*$; $b_Q = Q_0/\Delta S_0$; a, k, k_σ, n_σ, C_{00}, κ, constants; J_0, J_1, Y_0, Bessel functions; μ_1, α_n, roots of characteristic equations; j, current density; q, heat flux density; Po, Pe, Pomerantsev and Pecklet numbers, respectively; Nu, Nusselt number; Π_E, Π_U, Π_q, K_Q, K_S, similarity numbers; U, voltage; α, heat-transfer coefficient. Indexes: z, longitudinal component; *, boundary of conducting band; 0, base value; 1, value on the wall; 00, axial value; I, II, conducting and nonconducting bands, respectively.

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